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The Brooks–Jewett Theorem on an Orthomodular Lattice*

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The Brooks–Jewett convergence theorem for semigroup valued functions defined on an orthomodular lattice is proved. The proof is obtained by means of a lemma which allows the study of a sequence of finite additive and s-bounded functions defined on an orthomodular lattice with the Subsequential Interpolation Property through a sequence of σ-additive and s-bounded functions defined in a Boolean ring with the Subsequential Completeness Property. © 1991 Academic Press, Inc.

Previously many classical results of measure theory have been proved for functions defined on orthomodular posets or orthomodular lattices: the theory, in these cases, is sometimes described as "noncommutative measure theory"; [17, 19] give an organic overview of the results obtained for real valued functions. Recently some authors have proved noncommutative versions of convergence theorems [7, 11, 18, 9] and of decomposition theorems [17, 21, 20, 9], as well as for semigroup valued functions.

The main results of this paper are a Brooks–Jewett convergence theorem [2] and a uniform s-boundedness criterion of Cafiero [3] for semigroup valued functions defined on orthomodular lattices.

The theorems are proved by a key lemma (4.3) which allows us to study a sequence of finite additive and s-bounded functions, defined on an orthomodular lattice with the Subsequential Interpolation Property, by means of a sequence of σ-additive and s-bounded functions, defined in a Boolean ring with the Subsequential Completeness Property.

We also obtain a Brooks–Jewett theorem wich further generalizes those obtained recently by P. Morales [18], in the noncommutative theory, and those proved by D. Candeloro and G. Letta [4] in the Boolean theory.

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The paper is organized into six sections: in Section 1 the preliminary notions are presented, the theorems are proved in Section 5, and the key lemma is proved in Section 4. In Sections 2 and 3 some preliminary lemmas are established and in Section 6 we present some consequences of the main theorems.

1

We say that a pair $(P, \leqslant)$ is a *poset* if P is a nonempty set and $\leqslant$ is a partial ordering of P. In the following we employ the usual notations to indicate the supremum or the infimum of a subset of P, if such exists (cf. [16, 19]). A poset P may contain a smallest element and a largest one written as 0 and 1.

Let $(P, \leqslant)$ be a poset with 0 and 1; an *orthocomplementation* on P is a unary operation $'$ on P idempotent, decreasing, and such that $x \wedge x' = 0$ for all $x \in P$.

An *orthomodular poset* is a poset with 0, 1, and an orthocomplementation $'$ such that

(i) if $x, y \in P$ and $x \leqslant y'$ then $x \vee y$ exists,

(ii) if $x, y \in P$ and $x \leqslant y$ then $y = x \vee (y \wedge x')$.

An *orthomodular lattice* is an orthomodular poset which is also a lattice.

Let P be an orthomodular poset. Two elements x, y of P are said to be orthogonal, and we write $x \perp y$, when $x \leqslant y'$. A nonempty subset K of P is called *orthogonal* if any two different elements of K are orthogonal.

An orthomodular poset P is called *σ-orthocomplete* if the supremum of every countable orthogonal subset of P exists.

For details on orthomodular posets and orthomodular lattices we refer to [1, 16, 19].

If J is an infinite set the symbols $\mathscr{P}(J)$, $\mathscr{I}(J)$, and $\mathscr{F}(J)$, denote respectively the set of all the subsets of J, the set of all the infinite subsets of J, and the set of all the finite subsets of J, and we also write $\omega = \{0, 1, 2, ...\}$.

We say that an orthomodular lattice L has the *Subsequential Interpolation Property* if for every countable orthogonal set K in L and every $K_0 \in \mathscr{I}(K)$ there exist $b \in L$ and $H \in \mathscr{I}(K_0)$ such that

$$a \leqslant b \quad \forall a \in H, \qquad a \leqslant b' \quad \forall a \in K \setminus H,$$

in particular if $(a_i)_{i \in \omega}$ is an orthogonal sequence of L for every $M \in \mathscr{I}(\omega)$ there exist $b \in L$ and $N \in \mathscr{I}(M)$ such that

$$a_i \leqslant b, \quad \forall i \in N, \qquad a_i \leqslant b' \quad \forall i \in \omega \setminus N.$$

We say that an orthomodular lattice L has the *Subsequential Completeness Property* if for every orthogonal sequence $(a_i)_{i \in \omega}$ in L there exists $M \in \mathscr{I}(\omega)$ such that there exists the supremum of $\{a_i : i \in M\}$.

We observe that for a Boolean ring the Subsequential Completeness Property has been introduced by Constantinescu [5, 6] and Haydon [15] and the Subsequential Interpolation Property was introduced by Freniche [13] and Weber [23].

We observe that if R is a Boolean algebra with the Subsequential Interpolation Property (resp. with the Subsequential Completeness Property) and H is an orthomodular σ-orthocomplete lattice, the direct product $R \times H$ is an orthomodular lattice with the Subsequential Interpolation Property (resp. with the Subsequential Completeness Property). It is also possible to construct other examples by the Greechie Method [14], exposed also by Kalmbach [16, pp. 42, 43] and Rüttimann [19, 1.6], with an initial system of Boolean algebras with the Subsequential Interpolation Property (resp. with the Subsequential Completeness Property).

Let $(S, +, 0)$ be a commutative semigroup and let $\mathscr{U}$ be a uniformity on S. We say that $(S, +, 0, \mathscr{U})$ is a uniform semigroup if the function

$$(x, y) \rightarrow x + y$$

from $S \times S$ into S is uniformly continuous. In this case the uniformity can be generated by a set D of continuous pseudometrics d on S such that $d(x+z, y+z) \leqslant d(x, y)$ for every $x, y, z \in S$, (semi-invariant property).

For details on uniform semigroups we refer to Fox and Morales [12] and Weber [22].

2

Let $L = (L, \leqslant, 0, 1, ')$ be an orthomodular poset, let $S = (S, +, 0, \mathscr{U})$ an Hausdorff uniform semigroup. A function $\mu: L \rightarrow S$ is called *additive* if $\mu(0) = 0$ and $\mu(x \vee y) = \mu(x) + \mu(y)$ when $x, y \in L$, and $x \perp y$; an additive function μ is called *s-bounded* (resp. *σ-additive*) if for every orthogonal sequence $(a_i)_{i \in \omega}$ in L (resp. for every orthogonal sequence $(a_i)_{i \in \omega}$ in L such that there exists $\bigvee \{a_i : i \in \omega\}$) we have $\lim_i \mu(a_i) = 0$ (resp. $\mu(\bigvee \{a_i : i \in \omega\}) = \sum_{i \in \omega} \mu(a_i)$).

In the following we denote by $a(L, S)$, $sa(L, S)$, $ca(L, S)$, respectively, the set of additive or s-bounded or σ-additive functions on L to S.

A nonempty subset K of $sa(L, S)$ is said to be *uniformly s-bounded* if for every orthogonal sequence $(a_i)_{i \in \omega}$ in L we have $\lim_i \mu(a_i) = 0$ uniformly in $\mu \in K$.

The following Lemmas are proved in a straightforward manner, as in the Boolean case:

(2.1). *Let S be a complete uniform semigroup, L an orthomodular poset, and $\mu \in sa(L, S)$. If $(a_i)_{i \in \omega}$ is an orthogonal sequence in L then $(\mu(a_i))_{i \in \omega}$ is summable.*

(2.2). *Let S be a uniform semigroup, L a σ-orthocomplete orthomodular poset, and $(\mu_n)_{n \in \omega}$ a sequence in $ca(L, S)$ such that $\{\mu_n : n \in \omega\}$ is uniformly s-bounded. If $(a_i)_{i \in \omega}$ is an orthogonal sequence in L then $(\mu_n(a_i))_{i \in \omega}$ is summable uniformly in $n \in \omega$.*

(2.3). *Let S be a uniform semigroup, L an orthomodular lattice, and $\mu \in a(L, S)$. Then μ is s-bounded if and only if for every orthogonal sequence $(a_i)_{i \in \omega}$ in L, $\lim_i \mu(x \wedge a_i) = 0$ uniformly on $x \in L$.*

(2.4). *Let L be an orthomodular lattice with the Subsequential Interpolation Property, S a uniform semigroup, and $\mu \in sa(L, S)$. If $(a_i)_{i \in \omega}$ is an orthogonal sequence in L, for every neighborhood V of zero in S and every $M \in \mathscr{I}(\omega)$ there exist $N \in \mathscr{I}(M)$ and $b \in L$ such that*

$$a_i \leqslant b \quad \forall i \in N, \qquad a_i \leqslant b' \quad \forall i \in \omega \backslash N,$$

and

$$\mu(x \wedge b) \in V \qquad \forall x \in L.$$

Proof. Let V be a neighborhood of 0 and let $(M_k)_{k \in \omega}$ be a disjoint sequence of infinite subsets of M. By the Subsequential Interpolation Property, for every $k \in \omega$ there exist $b_k \in L$ and $N_k \in \mathscr{I}(M_k)$ such that

$$a_i \leqslant b_k \quad \forall i \in N_k, \qquad a_i \leqslant b'_k \quad \forall i \in \omega \backslash N_k.$$

Let $c_0 = b_0$ and, for every $k \in \omega \backslash \{0\}$, $c_k = b_k \wedge (\bigwedge \{b'_i : i = 0, 1, ..., k-1\})$ then $(c_k)_{k \in \omega}$ is an orthogonal sequence in L and, for (2.3), there exists $h \in \omega$ such that $\mu(x \wedge c_k) \in V$ for every $x \in L$. Since also

$$a_i \leqslant c_h \quad \forall i \in N_h, \qquad a_i \leqslant c'_h \quad \forall i \in \omega \backslash N_h,$$

c_h and N_h verify the conditions of the lemma.

3

Now we prove some useful Lemmas.

(3.1). *Let S be a complete uniform semigroup, $(x_i)_{i\in\omega}$ a summable sequence in S, and $(I_k)_{k\in K}$ a partition of ω such that, for every $k\in K$, $(x_i)_{i\in I_k}$ is summable. Then $(\sum_{i\in I_k} x_i)_{k\in K}$ is summable and*

$$\sum_{k\in K}\left(\sum_{i\in I_k} x_i\right)=\sum_{i\in\omega} x_i.$$

Proof. Let

$$s_k=\sum_{i\in I_k} x_i \quad \forall k\in K, \qquad \lambda=\sum_{i\in\omega} x_i.$$

If d is one of the semi-invariant pseudometrics which generate the uniformity of S and ε is a positive real number, let I_0 be an element of $\mathscr{I}(\omega)$ such that

$$d\left(\lambda, \sum_{i\in J} x_i\right)<\varepsilon$$

for every finite subset J of ω including I_0, and for every $k\in K$ let $I_0^{(k)}$ be an element of $\mathscr{I}(I_k)$ such that

$$d\left(s_k, \sum_{i\in J_k} x_i\right)<\varepsilon/2^k$$

for every finite subset J_k of I_k including $I_0^{(k)}$.

We write $H_0=\{k\in K: I_k\cap I_0\neq\varnothing\}$ and we suppose, without less of generality, $I_0^{(k)}$ including $I_k\cap I_0$ for every $k\in H_0$.

If H is a finite subset of K, including $\bigcup\{I_0^{(k)}: k\in H_0\}$, we have

$$d\left(\lambda, \sum_{k\in H} s_k\right)\leqslant d\left(\lambda, \sum_{i\in\bigcup_{k\in H} J_k} x_i\right)+\sum_{k\in H} d\left(\sum_{i\in J_k} x_i, s_k\right)\leqslant 3\varepsilon.$$

(3.2). *Let L be an orthomodular lattice with the Subsequential Interpolation Property, N an infinite subset of ω, and $(a_i)_{i\in\omega}$ an orthogonal sequence in L. If $\mathscr{G}$ is the subset of $\mathscr{P}(\omega)$ defined by the properties*

(i) $\Delta\in\mathscr{P}(N)$,

(ii) $\exists b\in L$ *such that*

$$a_i\leqslant b \quad \forall i\in\Delta, \qquad a_i\leqslant b' \quad \forall i\in N\setminus\Delta;$$

then $\mathscr{G}$ is a subring of $\mathscr{P}(\omega)$ with the Subsequential Completeness Property and

$$\{i\}\in\mathscr{G} \qquad \forall i\in N.$$

Proof. It is easy to prove that $\mathscr{G}$ is a ring. In order to prove that $\mathscr{G}$ has the Subsequential Completeness Property let $(\Delta_p)_{p\in\omega}$ be a disjoint sequence in $\mathscr{G}$ and $(b_p)_{p\in\omega}$ a sequence in L such that

$$a_i \leqslant b_p \quad \forall i \in \Delta_p, \qquad a_i \leqslant b'_p \quad \forall i \in \omega \backslash \Delta_p$$

for every $p \in \omega$.

Let $c_0 = b_0$, $c_p = b_p \wedge (\bigwedge \{b'_j : j = 1, ..., p-1\})$ for every $p \in \omega \backslash \{0\}$; then, for every $p \in \omega$

$$a_i \leqslant c_p \qquad \forall i \in \Delta_p. \tag{1}$$

Consider the orthogonal and countable subset of L

$$K = \{c_p : p \in \omega\} \cup \left\{a_i : i \in \omega \middle\backslash \bigcup_{p\in\omega} \Delta_p \right\}$$

and the infinite subset $K_0 = \{c_p : p \in \omega\}$. By the Subsequential Interpolation Property there exist $H \in \mathscr{I}(K_0)$ and $b \in L$ such that

$$a \leqslant b \qquad \forall a \in H, \qquad a \leqslant b' \qquad \forall a \in K \backslash H;$$

that is, there exists $M \in \mathscr{I}(\omega)$ such that

$$c_p \leqslant b \qquad \forall p \in M$$

$$c_p \leqslant b' \quad \forall p \in \omega \backslash M, \qquad a_i \leqslant b' \quad \forall i \in \omega \middle\backslash \bigcup_{p\in\omega} \Delta_p. \tag{2}$$

We prove that $\bigcup \{\Delta_p : p \in M\} \in \mathscr{G}$.

From (2) we have

$$a_i \leqslant b' \qquad \forall i \in \omega \middle\backslash \bigcup_{p\in\omega} \Delta_p.$$

If $i \in \bigcup \{\Delta_p : p \in \omega\}$ there exists a unique $q \in \omega$ such that $i \in \Delta_q$. Then, from (1) there exists a unique $q \in \omega$ such that

$$i \in \Delta_q, \qquad a_i \leqslant c_q. \tag{3}$$

From (2) and (3) we have $a_i \leqslant b$ if $i \in \Delta_q$ with $q \in M$ and $a_i \leqslant b'$ if $i \in \Delta_q$ with $q \in \omega \backslash M$. That is

$$a_i \leqslant b \quad \forall i \in \bigcup_{p\in M} \Delta_p, \qquad a_i \leqslant b' \quad \forall i \in \bigcup_{p\in\omega\backslash M} \Delta_p$$

and the proof is complete.

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From (2.1) and (3.1) we have

(4.1). *Let L be an orthomodular lattice, S a complete metric semigroup, and $\mu \in sa(L, S)$. For every orthogonal sequence $(a_i)_{i\in\omega}$ in L, the function*

$$\lambda : \Delta \in \mathscr{P}(\omega) \to \sum_{i\in\Delta} \mu(a_i) \in S$$

is σ-additive.

(4.2). *Let L be an orthomodular lattice, (S, d) a complete metric semigroup, and $(\mu_n)_{n\in\omega}$ a sequence in $sa(L, S)$. If $(a_i)_{i\in\omega}$ is an orthogonal sequence in L, there exist a decreasing sequence $(d_i)_{i\in\omega}$ in L and a decreasing sequence $(N_i)_{i\in\omega}$ in $\mathscr{I}(\omega)$ such that, for every $i \in \omega$,*

$$a_j \leqslant d_i \quad \forall j \in N_i, \qquad a_j \leqslant d_i' \quad \forall j \in \omega\backslash N_i$$
$$N_i \subseteq N_{i-1}\backslash\{\min N_{i-1}\}, \qquad (\textit{with} \quad N_{-1} = \omega)$$
$$d(\mu_p(x \wedge d_i), 0) < 1/i+1 \qquad \forall p \leqslant i, \quad \forall x \in L.$$

Proof. If $M = \omega\backslash\{0\}$, from (2.4) we have that there exist $b_0 \in L$ and $N_0 \in \mathscr{I}(\omega)$ such that

$$N_0 \subseteq \omega\backslash\{0\}, \qquad d(\mu_0(x \wedge d_0), 0) < 1 \quad \forall x \in L.$$

Let $n \in \omega$ and suppose that the sequences were constructed up to n; if $M = N_n\backslash\{\min N_n\}$, by (2.4) there exist $N_{n+1} \in \mathscr{I}(M)$ and $b \in L$ such that

$$a_j \leqslant b \quad \forall j \in N_{n+1}, \qquad a_j \leqslant b' \quad \forall j \in \omega\backslash N_{n+1}$$
$$d(\mu_p(x \wedge b), 0) < 1/n+2 \qquad \forall p \leqslant n+1, \quad \forall x \in L.$$

Setting $d_{n+1} = b \wedge d_n$, we complete the proof.

(4.3). *Let L be an orthomodular lattice with the Subsequential Interpolation Property, let (S, d) be a complete metric semigroup, $(\mu_n)_{n\in\omega}$ a sequence in $sa(L, S)$, and $(a_i)_{i\in\omega}$ an orthogonal sequence in L. Then there exist a subsequence $(a_{m_i})_{i\in\omega}$ of $(a_i)_{i\in\omega}$ and an infinite subset N of ω such that*

(1) *The subset $\mathscr{G}$ of $\mathscr{P}(\omega)$ defined by the properties:*

(i) $\Delta \in \mathscr{P}(N)$

(ii) *$\exists b \in L$ such that*

$$a_{m_i} \leqslant b \quad \forall i \in \Delta, \qquad a_{m_i} \leqslant b' \quad \forall i \in N \backslash \Delta$$

is a subring of $\mathscr{P}(\omega)$ with the Subsequential Completeness Property.

(2) $\{i\} \in \mathscr{G}$ $\forall i \in N$.

(3) *For every $n \in \omega$ the function*

$$\lambda_n: \Delta \in \mathscr{G} \to \sum_{i \in \Delta} \mu_n(a_{m_i}) \in S$$

is σ-additive and s-bounded.

(4) *For every $\Delta \in \mathscr{G}$ there exists $c_\Delta \in L$ such that*

$$\lambda_n(\Delta) = \mu_n(c_\Delta) \qquad \forall n \in N.$$

Proof. Let $(d_i)_{i \in \omega}$ be a decreasing sequence in L and $(N_i)_{i \in \omega}$ a decreasing sequence in $\mathscr{I}(\omega)$ which verify the (4.2) lemma's conditions and let $m_i = \min N_i$ for every $i \in \omega$.

We consider the subsequence $(a_{m_i})_{i \in \omega}$ of $(a_i)_{i \in \omega}$ and we observe that, for every $i \in \omega$, we have

$$a_{m_i} \leqslant d_i \wedge d'_{i+1}, \qquad a_{m_j} \leqslant d'_i \vee d_{i+1} \quad \forall j \in \omega \backslash \{i\} \tag{1}$$

because

$$m_p \in N_i \quad \forall p \geqslant i, \qquad m_p \in \omega \backslash N_i \quad \forall p \leqslant i-1$$

and consequently

$$a_{m_p} \leqslant d_i \quad \forall p \geqslant i, \qquad a_{m_p} \leqslant d'_i \quad \forall p \leqslant i-1.$$

The set

$$\{a_{m_i} : i \in \omega\} \cup \{d_i \wedge d'_{i+1} \wedge a'_{m_i} : i \in \omega\}$$

is a countable orthogonal subset of L; then for the Subsequential Interpolation Property there exist $N \in \mathscr{I}(\omega)$ and $a \in L$ such that

$$a_{m_i} \leqslant a \quad \forall i \in N, \qquad a_{m_i} \leqslant a' \quad \forall i \in \omega \backslash N$$
$$d_i \wedge d'_{i+1} \wedge a'_{m_i} \leqslant a' \qquad \forall i \in \omega.$$

The subset $\mathscr{G}$ of $\mathscr{P}(\omega)$ defined by (i) and (ii) of (1) is, for (3.2), a Boolean ring with the Subsequential Completeness Property such that

$$\{i\} \in \mathscr{G} \qquad \forall i \in N.$$

By (4.1), the function

$$\gamma_n: \Delta \in \mathscr{P}(\omega) \to \sum_{i \in \Delta} \mu_n(a_{m_i})$$

belongs to $ca(\mathscr{P}(\omega), S)$, for every $n \in \omega$; we prove that the restriction λ_n of γ_n to $\mathscr{G}$ is also s-bounded.

If $(\Delta_k)_{k \in \omega}$ is a disjoint sequence in $\mathscr{G}$ then $(b_k)_{k \in \omega}$ is a sequence in L such that

$$a_{m_i} \leqslant b_k \quad \forall i \in \Delta_k, \qquad a_{m_i} \leqslant b'_k \quad \forall i \in N \backslash \Delta_k$$

and, if we write

$$c_0 = b_0, \qquad c_k = b_k \wedge \left(\bigwedge \{b'_i : i = 0, ..., k-1\} \right) \quad \forall k \in \omega \backslash \{0\},$$

$(c_k)_{k \in \omega}$ is an orthogonal sequence in L such that, for every $k \in \omega$,

$$a_{m_i} \leqslant c_k \qquad \forall i \in \Delta_k.$$

By the s-boundedness of μ_n and by (2.3) we have, then, that if V is a closed neighbourhood of zero there exists $h \in \omega$ such that

$$\mu_n(x \wedge c_k) \in V \qquad \forall k \geqslant h \qquad \text{and} \qquad \forall x \in L;$$

then, for $k \geqslant h$,

$$\begin{aligned} \lambda_n(\Delta_k) &= \lim_{F \in \mathscr{F}(\Delta_k)} \sum_{i \in F} \mu_n(a_{m_i}) \\ &= \lim_{F \in \mathscr{F}(\Delta_k)} \mu_n \left(\bigvee \{a_{m_i} : i \in F\} \right) \\ &= \lim_{F \in \mathscr{F}(\Delta_k)} \mu_n \left(c_k \wedge \left(\bigvee \{a_{m_i} : i \in F\} \right) \right) \in V. \end{aligned}$$

In order to prove (4), choosing $\Delta \in \mathscr{G}$ there exists $b \in L$ such that

$$a_{m_i} \leqslant b \quad \forall i \in \Delta, \qquad a_{m_i} \leqslant b' \quad \forall i \in N \backslash \Delta,$$

and we show that

$$\lambda_n(\Delta) = \mu_n(d_0 \wedge a \wedge b) \qquad \forall n \in N.$$

The sequence $(d_i)_{i \in \omega}$ is decreasing; hence, for every $q \in \omega$ we have, for (1),

$$\begin{aligned} d_0 &= \left(\bigvee \{ d_i \wedge d'_{i+1} : i = 0, ..., q-1 \} \right) \vee d_q \\ &= \left(\bigvee \{ d_i \wedge d'_{i+1} \wedge a'_{m_i} : i = 0, ..., q-1 \} \right) \\ &\quad \vee \left(\bigvee \{ a_{m_i} : i = 0, ..., q-1 \} \right) \vee d_q \\ &= A \vee B, \end{aligned}$$

where

$$A = \left(\bigvee \{ d_i \wedge d'_{i+1} \wedge a'_{m_i} : i = 0, ..., q-1 \} \right)$$

$$\vee \left(\bigvee \{ a_{m_i} : i = 0, ..., q-1, i \notin \Delta \} \right)$$

$$B = \left(\bigvee \{ a_{m_i} : i = 0, ..., q-1, i \in \Delta \} \right) \vee d_q.$$

Since A is orthogonal to $a \wedge b$ and B, we have, focusing on A (cf. [1, Theorem II.3.10; or 16, Chap. 1, Theorem 3.5]

$$\begin{aligned} a \wedge b \wedge d_0 &= a \wedge b \wedge (A \vee B) \\ &= a \wedge b \wedge \left(\left(\bigvee \{ a_{m_i} : i = 0, ..., q-1, i \in \Delta \} \right) \vee d_q \right). \end{aligned}$$

Moreover $\bigvee \{ a_{m_i} : i = 0, ..., q-1, i \in \Delta \}$ is orthogonal to d_q and, from $\Delta \subseteq N$,

$$\{ a_{m_i} : i = 0, ..., q-1, i \in \Delta \} \leqslant a \wedge b,$$

follows. So focusing on $\bigvee \{ a_{m_i} : i = 0, ..., q-1, i \in \Delta \}$, we have

$$a \wedge b \wedge d_0 = \left(\bigvee \{ a_{m_i} : i = 0, ..., q-1, i \in \Delta \} \right) \vee (a \wedge b \wedge d_q)$$

and hence, for every $n \in \omega$,

$$\mu_n(a \wedge b \wedge d_0) = \sum_{i < q, i \in \Delta} \mu_n(a_{m_i}) + \mu_n(a \wedge b \wedge d_q).$$

For $\varepsilon > 0$ and $n \in \omega$, let $\bar{q} > n$ such that

$$1/q < \varepsilon, \qquad d\left(\sum_{i < q, i \in \Delta} \mu_n(a_{m_i}), \gamma_n(\Delta) \right) < \varepsilon \quad \forall q \geqslant \bar{q}$$

and also, with $n < q$,

$$d(\mu_n(x \wedge d_q), 0) < 1/q + 1 < \varepsilon \qquad \forall x \in L,$$

particularly

$$d(\mu_n(a \wedge b \wedge d_q), 0) < \varepsilon.$$

Then we have, for every $q > \bar{q}$,

$$\begin{aligned} &d(\mu_n(a \wedge b \wedge d_0), \gamma_n(\Delta)) \\ &\quad = d\left(\sum_{i < q, i \in \Delta} \mu_n(a_{m_i}) + \mu_n(a \wedge b \wedge d_q), \gamma_n(\Delta) \right) \\ &\quad \leqslant d\left(\sum_{i < q, i \in \Delta} \mu_n(a_{m_i}), \gamma_n(\Delta) \right) + d(\mu_n(a \wedge b \wedge d_q), 0) < 2\varepsilon. \end{aligned}$$

This completes the proof of the lemma.

5

We are now able to prove the main results.

(5.1). THE BROOKS–JEWETT THEOREM. *Let L be an orthomodular lattice with the Subsequential Interpolation Property, let S be a Hausdorff uniform semigroup, and $(\mu_n)_{n \in \omega}$ a sequence in $sa(L, S)$. If*

$$\lim_n \mu_n(a) = \mu_0(a)$$

for every $a \in L$, then $\{\mu_n : n \in \omega\}$ is uniformly s-bounded.

Proof. Suppose the contrary. Then we may assume, by passing to a subsequence if necessary, that there exist $d \in D$, $\varepsilon > 0$, and an orthogonal sequence $(a_i)_{i \in \omega}$ in L such that

$$d(\mu_i(a_i), 0) > \varepsilon \qquad \forall i \in \omega. \tag{1}$$

Consider the equivalence relation on S:

$$x \sim y \text{ if and only if } d(x, y) = 0;$$

then the set of all equivalence classes $\tilde{S} = S/\sim$ becomes a semigroup and $\tilde{d}([x], [y]) = d(x, y)$, where $[x]$ and $[y]$ belong to $S/\sim$, is a semi-invariant metric on $S/\sim$. Hence $(\tilde{S}, \tilde{d})$ is a metric semigroup and we write

(S_0, d_0), the completion. We observe, moreover, that the canonical projection π of (S, d) in (S_0, d_0) is continuous.

Let $\nu_n = \pi \circ \mu_n$ for every $n \in \omega$, $(\nu_n)_{n \in \omega}$ is a sequence in $sa(L, S_0)$, and it is

$$\lim_n \nu_n(a) = \nu_0(a) \qquad \forall a \in L.$$

From (4.3) we have that there exist a subsequence $(a_{m_i})_{i \in \omega}$ of $(a_i)_{i \in \omega}$, an infinite subset N of ω, and a subring $\mathscr{G}$ of $\mathscr{P}(\omega)$ with the Subsequential Completeness Property such that

(i) $\{i\} \in \mathscr{G}$ $\forall i \in N$

(ii) the functions

$$\lambda_n : \Delta \in \mathscr{G} \to \sum_{i \in \Delta} \nu_n(a_{m_i}) \in S_0, \qquad n \in \omega,$$

are σ-additives and s-bounded and

$$\lim_n \lambda_n(\Delta) = \lambda_0(\Delta)$$

for every Δ in $\mathscr{G}$.

It is well known that in a Boolean ring with the Subsequential Completeness Property the Brooks–Jewett theorem holds (cf. [10, Theorem (2.1); or 23, n.4 and n.7]. Hence $\{\lambda_n : n \in \omega\}$ is uniformly s-bounded. If $(i_k)_{k \in \omega}$ is an increasing sequence in N, $(\{i_k\})_{k \in \omega}$ is a disjoint sequence in $\mathscr{G}$; then we have

$$\lim_k \lambda_n(\{i_k\}) = 0$$

uniformly in $n \in \omega$.

Therefore we have for some $k \in \omega$,

$$d(\mu_n(a_{m_{i_k}}), 0) = d_0(\nu_n(a_{m_{i_k}}), 0) = d_0(\lambda_n(\{i_k\}), 0) < \varepsilon, \qquad \forall n \in \omega,$$

a contradiction of (1).

(5.2). Cafiero Theorem. *Let L be an orthomodular lattice with the Subsequential Interpolation Property, let S be a uniform Hausdorff semigroup, and $(\mu_n)_{n \in \omega}$ a sequence in $sa(L, S)$. Then $\{\mu_n : n \in \omega\}$ is uniformly s-bounded if and only if for every orthogonal sequence $(a_i)_{i \in \omega}$ in L and for every neighbourhood V of the zero of S there exist $\rho \in \omega$ and $\nu \in \omega$ such that*

$$\mu_n(a_\rho) \in V, \qquad \forall n \geqslant \nu.$$

Proof. The necessity of the condition is trivial.

For the sufficiency we observe that the Cafiero Theorem is true in a Boolean ring with the Subsequential Completeness Property. In order to prove that we can repeat the proof of [8, Prop. (1.3)], only using the Subsequential Completeness Property for a obvious modification (see also [23, Corollary 4.3, oss. p. 272 and oss. p. 273]).

That said, we can procede as in (5.1).

Suppose that G is an Abelian Hausdorff topological group; proceding as in the proof of (5.1) but with the use of Corollary (2.3) of [10] instead of Theorem (2.1), we can prove that if $(\mu_n)_{n\in\omega\setminus\{0\}}$ is a sequence in $sa(L, G)$ pointwise convergent to $\mu_0 \in a(L, G)$ then $\{\mu_n : n \in \omega\setminus\{0\}\}$ is uniformly s-bounded and hence $\mu_0 \in sa(L, G)$. We have then the group valued version of the Brooks–Jewett theorem:

(5.3). *Let L be an orthomodular lattice with the Subsequential Interpolation Property and let G be an Abelian Hausdorff topological group. If $(\mu_n)_{n\in\omega\setminus\{0\}}$ is a sequence in $sa(L, G)$ such that*

$$\lim_n \mu_n(a) = \mu_0(a) \qquad \forall a \in L$$

then $\{\mu_n : n \in \omega\}$ is uniformly s-bounded.

6

We now indicate some consequences of the theorems of Section 5.

(6.1) [17]. *Let P be a σ-orthocomplete orthomodular poset, S a uniform Hausdorff semigroup, and $(\mu_n)_{n\in\omega}$ a sequence in $sa(P, S)$ such that*

$$\lim_n \mu_n(a) = \mu_0(a) \qquad \forall a \in P;$$

then $\{\mu_n : n \in \omega\}$ is uniformly s-bounded.

Proof. Let $(a_i)_{i\in\omega}$ be an orthogonal sequence in P; we write

$$H = \{a_i : i \in \omega\} \cup \left\{\bigwedge \{a_i' : i \in \omega\}\right\}.$$

Then

$$L = \left\{\bigvee \{a : a \in T\} : T \in \mathscr{P}(H)\right\}$$

is a σ-orthocomplete orthomodular sublattice of P. By (5.1) we have, hence,

$$\lim_i \mu_n(a_i) = 0$$

uniformly in $n \in \omega$.

Let G be an Abelian Hausdorff topological group, A a Boolean algebra, R a sublattice of A, and μ an element of $a(A, G)$. We say that μ is *R-s-bounded* if for every disjoint sequence $(a_i)_{i \in \omega}$ of R we have

$$\lim_i \mu(a_i) = 0.$$

If $(\mu_n)_{n \in \omega}$ is a sequence in $a(A, G)$, the *R-uniformly-s-boundedness* of $\{\mu_n : n \in \omega\}$ is defined similarly.

In a straightforward manner it is possible to prove that

(6.2). *If $(\mu_n)_{n \in \omega}$ is a sequence in $a(A, G)$, $\{\mu_n : n \in \omega\}$ is R-uniformly-s-bounded if and only if for every disjoint sequence $(a_i)_{i \in \omega}$ in R there exists $(a_{i_k})_{k \in \omega}$ such that*

$$\lim_k \mu_n(a_{i_k}) = 0$$

uniformly in $n \in \omega$.

Following [4] we say that R has the *(E) property* if for every disjoint sequence $(a_i)_{i \in \omega}$ in R there exists a subsequence $(a_{i_k})_{k \in \omega}$ such that, for every $H \in \mathscr{P}(\omega)$, $\bigvee \{a_{i_k} : k \in H\}$ exists and belongs to R.

(6.3) [4]. *Let G be an Abelian Hausdorff topological group, A a Boolean algebra, R a sublattice of A with the (E) property, and $(\mu_n)_{n \in \omega}$ a sequence in $a(A, G)$ such that*

(i) *μ_n is R-s-bounded for every $n \in \omega$*

(ii) *$(\mu_n(a))_{n \in \omega}$ converges for every $a \in R$; then $\{\mu_n : n \in \omega\}$ is R-uniformly-s-bounded.*

Proof. Let $(a_i)_{i \in \omega}$ be a disjoint sequence in R and $(a_{i_k})_{k \in \omega}$ a subsequence such that $\bigvee \{a_{i_k} : k \in H\}$ exists and belongs to R, for every $H \in \mathscr{P}(\omega)$; then

$$L = \left\{ \bigvee \{a_{i_k} : k \in H\} : H \in \mathscr{P}(\omega) \right\}$$

is a σ-complete Boolean algebra and hence by (5.3)

$$\lim_k \mu_n(a_{i_k}) = 0$$

uniformly in $n \in \omega$. By (6.2) the proof is complete.

Let P be an orthomodular poset, G an Abelian Hausdorff topological group, and $(\mu_n)_{n\in\omega}$ a sequence in $ca(P, G)$. We say that $\{\mu_n : n\in\omega\}$ is *uniformly countably additive* if for every orthogonal sequence $(a_i)_{i\in\omega}$ in P such that $\bigvee \{a_i : i\in\omega\}$ exists,

$$\lim_k \sum_{i=0}^{k} \mu_n(a_i) = \mu_n\left(\bigvee \{a_i : i\in\omega\}\right)$$

uniformly in $n\in\omega$.

By (5.3) we obtain the following proposition which is proved in a different manner in [9]:

(6.4). *Let P be a σ-orthocomplete orthomodular poset and let G be an Abelian Hausdorff topological group. If $(\mu_n)_{n\in\omega\setminus\{0\}}$ is a sequence in $ca(P, G)$ such that*

$$\lim_n \mu_n(a) = \mu_0(a)$$

for every $a\in L$, then $\mu_0\in ca(P, G)$ and $\{\mu_n : n\in\omega\}$ is uniformly countably additive

Proof. G being a group, by the σ-orthocompleteness of L we have $ca(P, G)\subseteq sa(P, G)$. Let $(a_i)_{i\in\omega}$ be an orthogonal sequence in P; we write

$$H = \{a_i : i\in\omega\} \cup \left\{\bigwedge \{a_i' : i\in\omega\}\right\}.$$

Then

$$L = \left\{\bigvee \{a : a\in T\} : T\in\mathscr{P}(H)\right\}$$

is a σ-orthocomplete orthomodular sublattice of P and by (5.3) we have

$$\lim_i \mu_n(a_i) = 0$$

uniformly in $n\in\omega$. Hence $\{\mu_n : n\in\omega\}$ is uniformly s-bounded and by (2.2) $\{\mu_n : n\in\omega\setminus\{0\}\}$ is uniformly countably additive. Now it is easy to see that $\mu_0\in ca(P, G)$ and the proof is complete.

We observe that (6.4) contains Theorem 4 of [7] and Theorem 2.2 of [11].

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